Week 5

Chain Rule

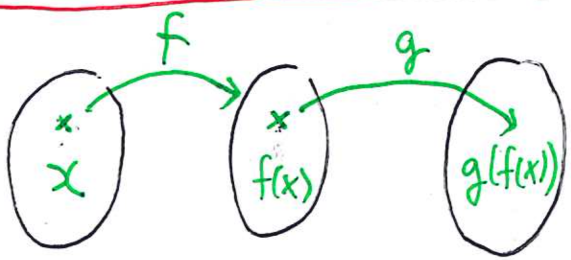
Let f be differentiable at a

g be differentiable at $f(a)$

Then $g \circ f$ is differentiable at a

$$\boxed{(g \circ f)'(x) = g'(f(x))\, f'(x)}$$

Rmk



$(g \circ f)(x) = g(f(x))$

f is called inner function

g is called outer function

Another form: Input x

Intermediate variable $u = f(x)$

Output $y = (g \circ f)(x) = g(f(x))$
$= g(u)$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

eg $\dfrac{d}{dx}(1+x+x^2)^{10}$

Sol. Let $f(x) = 1+x+x^2$ $\quad g(u) = u^{10}$

$\Rightarrow (1+x+x^2)^{10} = (g \circ f)(x) \qquad g'(u) = 10u^9$

$$\begin{aligned}\frac{d}{dx}(1+x+x^2)^{10} &= (g \circ f)'(x)\\ &= g'(f(x))\, f'(x)\\ &= 10(1+x+x^2)^9(1+2x)\end{aligned}$$

eg $\dfrac{d}{dx} \dfrac{1}{\sqrt{x^4+2x^2}}$

Sol let $u = x^4 + 2x^2$

$$y = \frac{1}{\sqrt{x^4+2x^2}} = \frac{1}{\sqrt{u}} = u^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \left(-\frac{1}{2} u^{-\frac{3}{2}}\right)(4x^3+4x)$$

$$= -\frac{1}{2}(x^4+2x^2)^{-\frac{3}{2}}(4x^3+4x)$$

$$= \frac{-2(x^3+x)}{(x^4+2x^2)^{\frac{3}{2}}}$$

Derivative of inverse

let f and f^{-1} be inverses

$y = f(x) \qquad x = f^{-1}(y)$

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[rr, bend right] & y \arrow[r, "f^{-1}"] & x \end{tikzcd}$$

$f^{-1} \circ f = \text{Identity}$

$$1 = \frac{dx}{dx} = \frac{dx}{dy} \cdot \frac{dy}{dx}$$

$$\Rightarrow \boxed{\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}}$$

$$\boxed{\begin{aligned}(f^{-1})'(y) &= \frac{1}{f'(x)} \\ &= \frac{1}{f'(f^{-1}(y))}\end{aligned}}$$

Trig. functions

$(\sin x)' = \cos x$ (1)

$(\cos x)' = -\sin x$ (2)

$(\tan x)' = \sec^2 x$

$(\sec x)' = \sec x \tan x$

$(\csc x)' = -\csc x \cot x$

$(\cot x)' = -\csc^2 x$

Rmk x is measured in radian, NOT degree

Pf of (1)

Formula:

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$(\sin x)' = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}}{h}$$

$$= \lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}$$

$$= \underbrace{\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)}_{\text{continuous in } h}\cdot\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}$$

$$= \cos\left(\frac{2x}{2}\right)(1)$$

$$= \cos x$$

Pf of (2)

Method I: Using

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Method II: $\cos x = \sin\left(\frac{\pi}{2}-x\right)$

Use Chain Rule: Let $u = \frac{\pi}{2}-x$, $y=\sin u$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = (\cos u)(0-1) = -\cos u$$

$$= -\cos\left(\frac{\pi}{2}-x\right) = -\sin x$$

The others

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

Pf of their derivatives:

Method I: from definition

Method II: Quotient rule and 1, 2

Ex Find $\frac{d}{dx}\tan(3x^2+1)$

Sol Let $u = 3x^2+1$ $y = \tan u$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = (\sec^2 u)(6x)$$

$$= \left(\sec^2(3x^2+1)\right)6x$$

Ex Let $f(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Find $f'(x)$.

Sol. For $a \neq 0$, $f(x) = x^2 \sin\frac{1}{x}$ near a

$$f'(x) = 2x\sin\frac{1}{x} + \underbrace{x^2\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}_{\text{Chain rule}}$$

For $x \neq 0$

$$= 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

For $x = 0$,

$$f'(0) = \lim_{h\to0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to0}\frac{h^2\sin\left(\frac{1}{h}\right) - 0}{h}$$

$-|h| \leq h\sin\left(\frac{1}{h}\right) \leq |h|$

Sandwich thm

$$= \lim_{h\to0} h\sin\left(\frac{1}{h}\right) = 0$$

Rmk: f is differentiable at every real number. However, $f'(x)$ is not continuous at 0

Exponential/Logarithm functions

Let $a>0$, $a\neq 1$ be a constant

$$(a^x)' = (\ln a)\,a^x \quad ③$$

$$(\log_a x)' = \frac{1}{x\ln a}$$

Special case: $a=e$

$$(e^x)' = e^x \quad ②$$

$$(\ln x)' = \frac{1}{x} \quad ①$$

Rmk $\ln x = \log_e x$

Pf of ① (Partial)

$$\lim_{h\to 0^+}\frac{\ln(x+h)-\ln(x)}{h}$$

$$= \lim_{h\to 0^+}\frac{1}{h}\ln\left(\frac{x+h}{x}\right)$$

$$= \lim_{h\to 0^+}\frac{1}{x}\frac{x}{h}\ln\left(1+\frac{h}{x}\right)$$

$$= \lim_{h\to 0^+}\frac{1}{x}\ln\left(1+\frac{1}{\frac{x}{h}}\right)^{\frac{x}{h}}$$

$\because \ln$ is continuous

$$= \frac{1}{x}\ln\left(\lim_{h\to 0^+}\left[\left(1+\frac{1}{\frac{x}{h}}\right)^{\frac{x}{h}}\right]\right)$$

$$= \frac{1}{x}\ln e$$

$$= \frac{1}{x}$$

Recall:

$$\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^y = e$$

As $h\to 0^+$

$$\frac{x}{h}\to +\infty$$

$$\Rightarrow \lim_{h\to 0^+}\left(1+\frac{1}{\frac{x}{h}}\right)^{\frac{x}{h}} = e$$

Ex Show that

$$\lim_{h\to 0^-}\frac{\ln(x+h)-\ln(x)}{h} = \frac{1}{x} \quad \text{(More difficult)}$$

Hint:

$$1+\frac{h}{x} = \frac{x+h}{x} = \frac{1}{1-\frac{h}{x+h}}$$

Pf of 2 from 1

let $y = e^x \quad x = \ln y$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$ (Chain rule)

$$= \frac{1}{\frac{1}{y}}$$

$$= y$$

$$= e^x$$

$$\Rightarrow (e^x)' = e^x$$

Pf. of 3

let $y = a^x = e^{\ln a^x} = e^{x \ln a}$

Let $u = x \ln a \qquad = e^u$

$$(a^x)' = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= e^u \cdot \ln a$$

$$= e^{x \ln a} \ln a$$

$$= (a^x) \ln a$$

$$\Rightarrow 3$$

Inverse of trig. functions

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \quad (1) \qquad (\text{arcsec}\, x)' = \frac{1}{|x|\sqrt{x^2-1}}$$

$$(\arccos x)' = \frac{-1}{\sqrt{1-x^2}} \qquad (\text{arccsc}\, x)' = \frac{-1}{|x|\sqrt{x^2-1}}$$

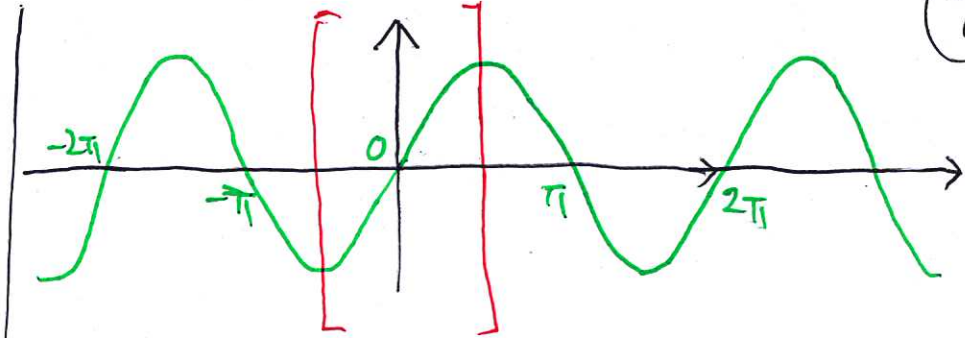
$$(\arctan x)' = \frac{1}{1+x^2} \qquad (\text{arccot}\, x)' = -\frac{1}{1+x^2}$$

They are inverses of trig fuction.

$\arcsin : [-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\arccos : [-1, 1] \longrightarrow [0, \pi]$

$\arctan : \mathbb{R} \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$



$\sin x$ is one to-one in $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Pf of (1)

let $y = \arcsin x \qquad x = \sin y$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y} \qquad \cos^2 y + \sin^2 y = 1$$

$$\cos y = \pm\sqrt{1-\sin^2 y}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}} \qquad \because -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$$= \frac{1}{\sqrt{1-x^2}} \qquad \Rightarrow \cos y \ge 0$$

Thm If f is differentiable at a, then f is continuous at a

Differentiable $\Rightarrow$ Continuous

Equivalent statement

Not Continuous $\Rightarrow$ Not Differentiable

Recall

Continuous at a: $\lim_{x \to a} f(x) = f(a)$

Differentiable at a: $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists

How to relate them?

Pf $\left(\lim_{x \to a} f(x)\right) - f(a)$ ...

$$= \lim_{x \to a} f(x) - f(a)$$

$$= \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \cdot (x-a)$$

$$= \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x \to a} x-a$$

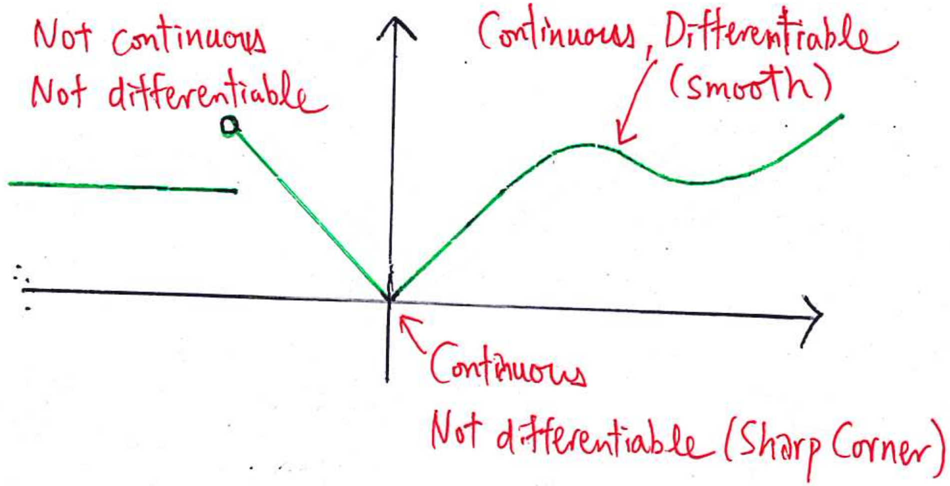
f is differentiable at a

$$= f'(a) \cdot (a-a)$$

$$= 0$$

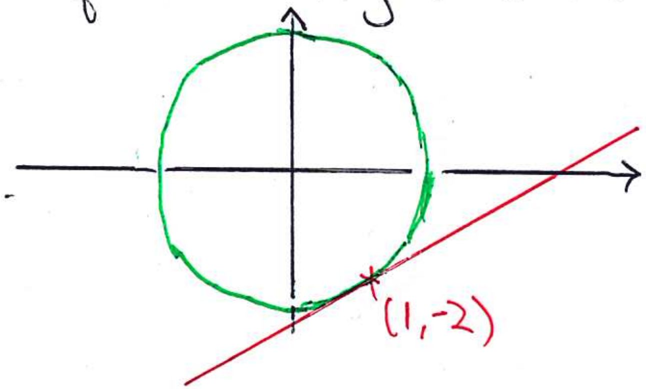
$\Rightarrow \lim_{x \to a} f(x) = f(a) \Rightarrow$ continuous at a

eg



Implicit differentiation

eg. Consider circle $\mathcal{C}: x^2+y^2=5$.
Find equation of tangent at $(1,-2)$



Sol Method I (Express y in terms of x)

$$x^2+y^2=5$$
$$y=\pm\sqrt{5-x^2}$$

Near $(1,-2)$

$$y<0 \Rightarrow y=-\sqrt{5-x^2}$$
$$=-(5-x^2)^{\frac{1}{2}}$$
$$\frac{dy}{dx}=-\left(\frac{1}{2}\right)(5-x^2)^{-\frac{1}{2}}(-2x)$$
$$=x(5-x^2)^{-\frac{1}{2}}$$

$$\left.\frac{dy}{dx}\right|_{x=1}=(1)(5-1^2)^{-\frac{1}{2}}=\frac{1}{2} \leftarrow \text{slope of tangent}$$

$\Rightarrow$ Equation of tangent: $y=\frac{1}{2}(x-1)-2$

Method II (Implicit differentation)

Idea: $x^2+y^2=5 \Rightarrow y$ depends on x
y can be regarded as a function $y(x)$ near $(1,-2)$.
We can find $\frac{dy}{dx}$ without explicitly finding $y(x)$.

Apply $\frac{d}{dx}$ to $x^2+y^2=5$

$$\Rightarrow \frac{d}{dx}(x^2)+\boxed{\frac{d}{dx}(y^2)}=\frac{d}{dx}(5)$$
$$2x+\boxed{2y\frac{dy}{dx}}=0$$

Chain rule
$$\frac{d}{dx}y^2=\left(\frac{d}{dy}y^2\right)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=-\frac{x}{y} \qquad \left.\frac{dy}{dx}\right|_{(1,-2)}=-\frac{1}{-2}=\frac{1}{2}$$

Same remaining steps as in Method I

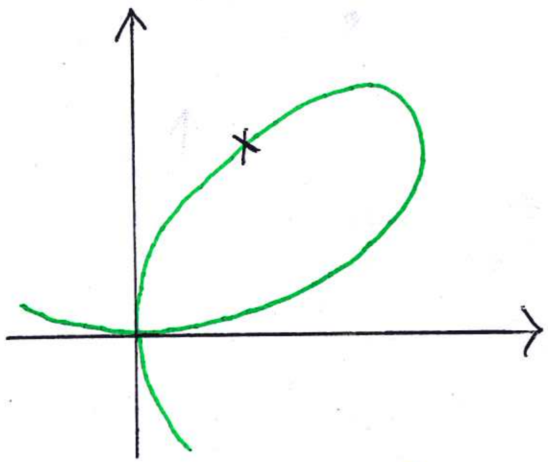
Exercise

Let $\mathscr{C}: x^3+y^3-9xy=0$

i. Show that $(2,4)$ is on $\mathscr{C}$

ii Find equation of tangent at $(2,4)$

Comment: It is hard to use method I of last example



Ans of ii: $y=\frac{4}{5}x+\frac{12}{5}$

Higher derivatives

For $n\geq 0$,

define $\dfrac{d^n y}{dx^n} = \underbrace{\dfrac{d}{dx}\left(\cdots\cdots \dfrac{d}{dx}\left(\dfrac{d}{dx}\right.\right.}_{\text{differentiate } n \text{ times}} \left.\left. y\right)\cdots\right)$ (called the n-th derivative)

Other notations: $\dfrac{d^n y}{dx^n} = y^{(n)} = f^{(n)}(x)$ if $y=f(x)$

Second derivative: $\dfrac{d^2y}{dx^2} = y^{(2)} = y'' = f''(x) = f^{(2)}(x)$

eg $y=x^2+3x+7$

$y'=2x+3$

$y''=2$

$y^{(n)}=0$ for $n\geq 3$

eg $f(x)=\sin x$. Then

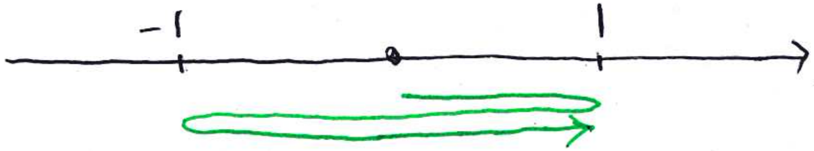
$$f^{(n)}(x)=\begin{cases} \sin x & \text{if } n=4m \\ \cos x & \text{if } n=4m+1 \\ -\sin x & \text{if } n=4m+2 \\ -\cos x & \text{if } n=4m+3 \end{cases}$$

$m\in\mathbb{Z}$

Physical meaning

Let $x(t)$ = displacement (position) of a particle at time t

eg $x(t) = \sin t$, then the particle is moving between 1 and -1 on the real line



Then derivatives of x are:

$v(t) = x'(t)$ = velocity

$a(t) = x''(t)$ = acceleration

eg Suppose $ye^x = \cos(2x+y-1)$

Find y' and y'' at $(x,y) = (0,1)$

Sol Differentiate the given equation (with respect to x)

$$y'e^x + ye^x = \left(-\sin(2x+y-1)\right)(2+y') \quad (*)$$

Differentiate once more

$$y''e^x + y'e^x + y'e^x + ye^x = \left(-\cos(2x+y-1)\right)(2+y')^2 + \left(-\sin(2x+y-1)\right)(y'') \quad (**)$$

Put $x=0, y=1$ into $(*)$

$$\Rightarrow y'\big|_{(0,1)} + 1 = 0 \Rightarrow \boxed{y'\big|_{(0,1)} = -1}$$

Put $x=0, y=1, y'=-1$ into $(**)$,

$$\Rightarrow y'' - 1 - 1 + 1 = (-1)(2-1)^2 + 0 \Rightarrow \boxed{y''\big|_{(0,1)} = 0}$$